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NONLINEAR FILTERING EQUATIONS FOR TWO-PARAMETER SEMIMARTINGALES

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Nonlinear filtering equations for two-parameter semimartingales of a Brownian sheet are obtained by an extended reference probability method, also applicable to the known Gaussian linear models.

Two-parameter semimartingale	nonlinear filtering
Brownian sheet	reference probability

1. Introduction and preliminaries

1.1. *Introduction*

Nonlinear filtering equations are given here for a model in which the signal X is a semimartingale of a Brownian sheet B, in the sense of Wong and Zakai [24] and the observation process Y is defined by

$$Y_{s,t}=\int_0^s\int_0^t H_{u,v}\,\mathrm{d}u\,\mathrm{d}v+W_{s,t}$$

where H is a nonanticipative functional of X and W is a Brownian sheet, independent of B.

For the formulation of the causal filtering problem considered here, we refer to Wong and Tsui [18] and to Wong [17]. In this formulation the rectangle $R_{s,t}=\{(u,v)\in\mathbb{R}_+^2\colon u\le s,\ v\le t\}$ describes a certain past and, according to the displacements,

$$(s,t)\to(s+\mathrm{d}s,t),\ (s,t)\to(s,t+\mathrm{d}t)\quad\text{and}\quad(s,t)\to(s+\mathrm{d}s,t+\mathrm{d}t);$$

three types of present are defined for $R_{s,t}$, namely, its east, north and north-east boundaries. The filtering equations concern the estimation of the present values of X in terms of the past values of Y, and each of the mentioned displacements of (s,t) yields to a filtering equation that we respectively call horizontal, vertical and diagonal.

In the one-parameter filtering theory there exist mainly two methods for the derivation of the nonlinear filtering equation. The first, developed by Fujisaki, Kallianpur and Kunita in [5], is the direct method based on the martingale representation in terms of the innovation process; whereas the second, initially suggested by Zakai in [25], is the reference probability method widely applied and developed later on by several authors. In the two-parameter case the direct method has been used by Wong and Tsui in [18], Wong in [17] and Korezlioglu in [8] for the Gaussian linear model. However, this direct method presents some theoretical difficulties because it necessitates the elaboration of a stochastic calculus with respect to a filtration not having the famous property F4 (Cairoli and Walsh [2]) by which means the most important results of the two-parameter martingale theory are obtained. As in our first work [11] we use here the reference probability method, extending the approaches of Bremaud and Yor [1] and Mazziotto and Szpirglas [15] in the one-parameter case. The method had already been used by Wong in [16] for the filtering of the Markovian solution of a two-parameter diffusion equation, such a process being a particular type of semimartingales considered here. The great advantage of the method is that the model can be defined on a reference probability space where the main filtrations have property F4; hence, all the results of the two-parameter martingale theory can be fully used. As shown in [11] and [15] the filtering equations are obtained by the combination of a projection theorem, a change of probability and Ito differentiation rules. These last two operations were rendered possible by the results of Wong and Zakai in [23] and [24].

In the second part of this section we recall a few notions of theory with some slight extensions needed for computations in the absence of property F4.

Section 2 is devoted to the description of the model and to the elaboration of the reference probability method. We construct the projection theorem for square-integrable semimartingales and obtain the desired filtering equations (here called unnormalized) on the reference probability space.

We derive the filtering equations in Section 3 under the general hypothesis that all the operations that we use are justified and the results are meaningful. By making such a hypothesis, merely justifying our formal computation, we wanted to see all the steps of the derivation of the filtering equations in order to be able to construct a set of hypotheses (on the model) under which the obtained results are valid. The diagonal filtering equation is presented in terms of conditional covariance functions extending the one given in [17] by Wong for the Gaussian linear case.

In Section 4 we give various hypotheses on the model justifying the results obtained in Section 3. In particular we show that the reference probability method can be adapted not only to the case where H is bounded as in [11] but also to more general situations amongst which the Gaussian models considered in [17, 8, 18].

Finally, in Section 5, we give as an example of application the filtering equations for a bidirectional diffusion process X studied in [10] by Korezlioglu and Mazziotto.

This case contains the Gaussian model of [17] for which we reproduce the filtering equations and the corresponding Riccati equations.

1.2. Preliminaries

1.2.1

We refer to Cairoli and Walsh [2] and Wong and Zakai [19–24] for preliminary notations, definitions and results, some of which are slightly extended for the needs of the paper.

Although many of the results obtained here are valid for processes indexed by $\mathbb{R}_+^2$, we only consider processes indexed by

$$R_{z_0} = \{(s,t) \in \mathbb{R}_+^2 : 0 \leq s \leq s_0, 0 \leq t \leq t_0\} \quad \text{with } z_0 = (s_0, t_0),$$

in order to make full use of the compactness of this set. We shall denote by z, x, y or (s,t), (a,b), (u,v) the generic points of R_{z_0}; s, a, u (resp. t, b, v) will always represent the horizontal (resp. vertical) coordinates. In order to shorten the notations, whenever a point (s,t) of $\mathbb{R}^2$ is used as an index, we shall write it as st: for instance X_{st} instead of $X_{(s,t)}$. The Borel σ-algebra of R_{z_0} is denoted by $\mathcal{R}_{z_0}$ and the Lebesgue measure on $\mathcal{R}_{z_0}$ by Λ. $\mathbb{R}^2$ is endowed with its usual partial order relation:

$$(s,t) < (u,v) \quad \text{iff} \quad s \leq u \text{ and } t \leq v.$$

$(s,t) \ll (u,v)$ stands for $s < u$, $t < v$; '$\not<$' (resp. '$\not\ll$') means 'not$<$' (resp. 'not$\ll$').

For $z \in R_{z_0}$, R_z denotes the rectangle $\{x : 0 < x < z\}$, i.e., the 'past parameters' in R_{z_0} up to the point z. We call horizontal (resp. vertical) present of R_{st} the line segment $\{s\} \times [0,t]$ (resp. $[0,s] \times \{t\}$) and denote it by $\partial^1 R_{st}$ (resp. $\partial^2 R_{st}$). The present of R_{st} is its north-east boundary, i.e., $\partial^1 R_{st} \cup \partial^2 R_{st}$ and it will be denoted by ∂R_{st}.

For two real numbers a_1 and a_2, $a_1 \vee a_2$ (resp. $a_1 \wedge a_2$) indicates their maximum (resp. minimum). For two points $x = (a,b)$ and $y = (u,v)$ of $\mathbb{R}^2$, $x \vee y$, $x \wedge y$ and $x \otimes y$ represent the points $(a \vee u, b \vee v)$, $(a \wedge u, b \wedge v)$ and (a, v), respectively; $x \curlywedge y$ (resp. $(x \overset{\wedge}{\curlywedge} y)$) stands for $a \leq u, b \geq v$ (resp. $a < u, b > v$). We denote by $I_{x,y}$ the indicator of the set $\{(x,y) \in R_{z_0}^2 : x \overset{\wedge}{\curlywedge} y\}$ and by $]x,y]$ the set $\{z : x \ll z < y\}$.

1.2.2

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space. A filtration $\mathcal{F}$ is a family $(\mathcal{F}_z, z \in R_{z_0})$ of sub-σ-algebras of $\mathcal{A}$ having the following properties:

F1: $x < y \Rightarrow \mathcal{F}_x \subset \mathcal{F}_y$ (*increasingness*),

F2: $\mathcal{F}_0$ contains all $\mathbf{P}$-negligible sets of $(\Omega, \mathcal{A})$ (*completeness*),

F3: $\forall z \in R_{z_0}$: $\mathcal{F}_z = \bigcap_{x \gg z} \mathcal{F}_x$ (*right continuity*).

For $z = (s,t)$, $\mathcal{F}_z^1$ or $\mathcal{F}_s^1$ (resp. $\mathcal{F}_z^2$ or $\mathcal{F}_t^2$) denotes the σ-algebra $\mathcal{F}_{st_0}$ (resp. $\mathcal{F}_{s_0t}$), and $\mathcal{F}^1$ (resp. $\mathcal{F}^2$) the one-parameter filtration $(\mathcal{F}_s^1, s \leq s_0)$ (resp. $(\mathcal{F}_t^2, t \leq t_0)$); $\mathcal{F}_z^*$ or $\mathcal{F}_{st}^*$ represents the σ-field $\mathcal{F}_s^1 \vee \mathcal{F}_t^2$. The following property is often needed for $\mathcal{F}$

in the two-parameter martingale theory:

F4: $\forall (s,t) \in R_{z_0}$; $\mathscr{F}^1_s$ and $\mathscr{F}^2_t$ are conditionally independent given $\mathscr{F}_{st}$.

Nevertheless, it often happens [23, 7], as in the filtering problem considered here, that this property is not preserved under a change of probability – hence the necessity of extending the definitions and properties of martingales and stochastic integrals with respect to filtrations not having property F4.

Although a process X and a filtration $\mathscr{F}$ are respectively denoted by $(X_z, z \in R_{z_0})$ and $(\mathscr{F}_z, z \in R_{z_0})$, we extend the index set to $\bar{R}_{z_0} = \{z \in \mathbb{R}^2, z < z_0\}$ taking for $z \not> 0$, X_z equal to 0 and $\mathscr{F}_z$ equal to the σ-algebra generated by all the **P**-negligible sets of $(\Omega, \mathscr{A})$. This will eliminate the privileged role of the coordinate axes of $\mathbb{R}^2_+$.

Given an $\mathbb{R}^n$-valued process X on $(\Omega, \mathscr{A}, \mathbf{P})$, the natural filtration of X, denoted by $\mathscr{F}^X$, is the smallest filtration to which the components of X (that we denote by $X^1, X^2, \ldots, X^n$) are adapted.

Unless otherwise stated, all the processes we consider here are real-valued processes. A process X is said to be *integrable* (resp. *square-integrable*, *p-integrable*) if, for all $z \in R_{z_0}$,

$$\mathbf{E}(|X_z|) < \infty \quad (\text{resp. } \mathbf{E}(X_z^2) < \infty, \mathbf{E}(|X_z|^p) < \infty \text{ for } p \geq 2).$$

The variation of a process X on a rectangle $]x, y]$ in $\bar{R}_{z_0}$ is defined by

$$X(]x, y]) = X_y - X_{y \wedge x} - X_{x \wedge y} + X_x.$$

For a separable process X, X^* stands for $\sup_{z < z_0} |X_z|$, and $H^p(\mathscr{F}, \mathbf{P})$, $p \geq 1$, denotes the space of all separable $\mathscr{F}$-adapted processes X such that X^* is p-integrable with respect to **P**.

1.2.3

Let $X = (X_z, z \in \bar{R}_{z_0})$ be a real-valued integrable process on $(\Omega, \mathscr{A}, \mathbf{P})$ adapted to a given filtration $\mathscr{F}$. Then X is said to be a *strong martingale* (resp. *weak martingale*), if, for all $z < z'$ in $\bar{R}_{z_0}$,

$$\mathbf{E}(X(]z, z'])/\mathscr{F}^*_z) = 0 \text{ a.s.} \quad (\text{resp. } \mathbf{E}(X(]z, z'])/\mathscr{F}_z) = 0 \text{ a.s.}),$$

and a martingale if, for all $z < z'$,

$$\mathbf{E}(X_{z'}/\mathscr{F}_z) = X_z \text{ a.s.}$$

An integrable process X is said to be a *1-martingale* (resp. *2-martingale*) if, for any fixed $t \in [0, t_0]$ (resp. $s \in [0, s_0]$), the one-parameter process $(X_{st}, s \in [0, s_0])$ (resp. $(X_{st}, t \in [0, t_0])$) is an $\mathscr{F}^1$-martingale (resp. $\mathscr{F}^2$-martingale).

A strong martingale is also a martingale, and an adapted 1- and 2-martingale, all of which are weak martingales. If property F4 holds, then X is a martingale iff it is an adapted 1- and 2-martingale.

When we need to underline the filtration $\mathscr{F}$ and the probability **P** for which X may have the above properties, we write $\mathscr{F}$-martingale, $(\mathscr{F}, \mathbf{P})$-martingale and so on.

1.2.4

A stochastic process W is called a *Brownian sheet* on R_{z_0}, if it is continuous (i.e., almost all trajectories are continuous on R_{z_0}) and if it is a centered Gaussian process whose covariance function is given by

$$\mathbf{E}(W_{st}W_{uv}) = (s \wedge u)(t \wedge v).$$

For a Borel set A in R_{z_0}, we denote by $\sigma(\mathrm{d}W_z, z \in A)$ the sub-σ-algebra generated by the set of r.v.'s $\int_A f_z \,\mathrm{d}W_z$, where f runs over the set of all bounded non-random Borel functions on R_{z_0}.

The Brownian sheet W is called an $\mathcal{F}$ (resp. $\mathcal{F}$-1, $\mathcal{F}$-2)-Brownian sheet if it is $\mathcal{F}$ (resp. $\mathcal{F}^1, \mathcal{F}^2$)-adapted and, for each $z=(s,t)$ in R_{z_0}, the σ-fields $\mathcal{F}_z^*$ and $\sigma(\mathrm{d}W_x, x \gg z)$ (resp. $\mathcal{F}_s^1$ and $\sigma(\mathrm{d}W_{uv}, u > s, 0 \leq v \leq t_0)$, $\mathcal{F}_t^2$ and $\sigma(\mathrm{d}W_{uv}, 0 \leq u \leq s_0, v > t)$ are independent. Such a process is obviously a strong martingale (resp. 1-martingale, 2-martingale). The Brownian sheet W is called an *$\mathcal{F}$-Brownian martingale* if it is $\mathcal{F}$-adapted and, for each z in R_{z_0}, $\mathcal{F}_z$ and $\sigma(\mathrm{d}W_x, x \not\prec z)$ are independent. Such a process is a martingale and both $\mathcal{F}$-1 and $\mathcal{F}$-2-Brownian sheet. Conversely, an adapted $\mathcal{F}$-1- and $\mathcal{F}$-2-Brownian sheet is also an $\mathcal{F}$-Brownian martingale.

An $\mathbb{R}^n$-valued process is called a Brownian sheet if its components are mutually independent Brownian sheets. The above definitions are trivially extended to an $\mathbb{R}^n$-valued Brownian sheet. Finally, we recall that, for an n-dimensional Brownian sheet W, the filtration $\mathcal{F}^W$ has property F4.

A Brownian sheet can be characterized in terms of one-parameter Brownian motions as follows.

Proposition 1.1. *A continuous, $\mathbb{R}^n$-valued process $W = ((W_z^1, \ldots, W_z^n), z \in R_{z_0})$ vanishing on the coordinate axes, is an $\mathcal{F}$-1-Brownian sheet iff $\forall m \in \mathbb{N}^*$ and $\forall (t_1, \ldots, t_{m+1})$ such that $0 \leq t_1 < t_2 < \cdots < t_{m+1} \leq t_0$, the process*

$$\tilde{W} = \{((W^i_{st_{k+1}} - W^i_{st_k})(t_{k+1} - t_k)^{-1/2}),\ i = 1, \ldots, n,\ k = 1, \ldots, m,\ s \leq s_0\}$$

is a one-parameter $n \times m$-dimensional $\mathcal{F}^1$-Brownian motion on $[0, s_0]$.

Remark. A similar characterization can be given for an $\mathcal{F}$-2-Brownian sheet.

Proof. As the necessity of the condition is obvious, we only prove its sufficiency.

Suppose that the given condition holds. Then, for all t on $]0, t_0]$, $(W_{st}t^{-1/2}, s \in [0, s_0])$ is an $\mathcal{F}^1$-Brownian motion. Hence, the components of W_{st} are mutually independent centered Gaussian r.v.'s and $\mathbf{E}((W^i_{st})^2) = st$. Moreover, for $s < s'$ the r.v.'s

$$\{(W^i_{s't_{k+1}} - W^i_{s't_k}) - (W^i_{st_{k+1}} - W^i_{st_k}),\ i = 1, \ldots, n,\ k = 1, \ldots, m\},$$

are mutually independent Gaussian r.v.'s, independent of $\mathcal{F}_s^1$. Consider the partitions $0 = s_1 < s_2 < \cdots < s_{p+1} = s_0$ and $0 = t_1 < t_2 < \cdots < t_{q+1} = t_0$. Then the rectangles

$$A_{kl} =](s_k, t_l), (s_{k+1}, t_{l+1})], \quad k = 1, \ldots, p, l = 1, \ldots, q$$

form a partition of R_{z_0}. Let B_{ikl} be an element of $\sigma(\mathrm{d}W_z^i, z \in A_{kl})$. By successive conditioning one can prove the equality

$$\mathbf{P}\left(\bigcap_{k=1}^{p} \bigcap_{l=1}^{q} \bigcap_{i=1}^{n} B_{ikl}\right) = \prod_{k=1}^{p} \prod_{l=1}^{q} \prod_{i=1}^{n} \mathbf{P}(B_{ikl}).$$

Therefore,

$$\{W^i(A_{kl}), \quad i = 1, \ldots, n, k = 1, \ldots, p, l = 1, \ldots, q\}$$

is a set of mutually independent centered Gaussian r.v.'s. Now, any finite linear combination $\sum_{i,k,l} a_{ikl} W^i_{s_k t_l}$ can be written as $\sum_{i,k,l} a'_{ikl} W^i(A_{kl})$. Hence, such linear combinations are centered Gaussian r.v.'s. Consequently, W is a centered Gaussian process.

Finally, we have for $(u, v) \curlywedge (s, t)$

$$\mathbf{E}(W^i_{uv} W^j_{st}) = \mathbf{E}(((W^i_{uv} - W^i_{ut}) + W^i_{ut}) W^j_{st}) = ut\delta_{ij}$$

and for $(u, v) < (s, t)$

$$\mathbf{E}(W^i_{uv} W^j_{st}) = \mathbf{E}(W^i_{uv}((W^j_{st} - W^j_{sv}) + W^j_{sv})) = uv\delta_{ij}$$

where δ_{ij} is the Kronecker symbol. This proves that

$$\mathbf{E}(W^i_{uv} W^j_{st}) = (u \wedge s)(v \wedge t)\delta_{ij}.$$

Therefore, W is a Brownian sheet. The independence of $\mathcal{F}_s^1$ and $\sigma(\mathrm{d}W_{uv}, u > s, 0 \leq v \leq t_0)$ is a trivial consequence of the hypothesis.

1.2.5

We denote by $\mathcal{H}_i^p(\mathcal{F})$, $i = 0, 1, 2$ and $p \in [1, \infty[$, the space of all processes $\phi = (\phi_z, z \in R_{z_0})$ satisfying the following conditions:

(a) ϕ is measurable with respect to $\mathcal{R}_{z_0} \otimes \mathcal{F}_{z_0}$,

(b_0) (resp. (b_1), (b_2)) ϕ is $\mathcal{F}$(resp. $\mathcal{F}^1$, $\mathcal{F}^2$)-adapted,

(c) $$\|\phi\|_{\mathcal{H}_i^p} = \left[\int_{R_{z_0}} \mathbf{E}(|\phi_z|^p)\,\mathrm{d}1_z\right]^{1/p} < \infty.$$

We denote by $\mathcal{H}^p(\mathcal{F})$, for integer p, the Banach space of processes Ψ, indexed by $R_{z_0}^2$: $\Psi = (\Psi_{xy}, (x, y) \in R_{z_0}^2)$ satisfying the following conditions:

(d) $\forall (x, y) \in R_{z_0}^2$, $\Psi_{xy} = 0$ unless $x \curlywedge y$,

(e) Ψ is measurable with respect to $\mathcal{R}_{z_0} \otimes \mathcal{R}_{z_0} \otimes \mathcal{F}_{z_0}$,

(f) $\forall (x, y) \in R_{z_0}^2$, Ψ_{xy} is $\mathcal{F}_{x \vee y}$-measurable,

(g) $$\|\Psi\|_{\mathcal{H}^p} = \left[\int_{R_{z_0}} \mathbf{E}(|\Psi_{xy}|^p)\,\mathrm{d}1_x\,\mathrm{d}1_y\right]^{1/p} < \infty$$

We denote by $\tilde{\mathscr{H}}_i^p(\mathscr{F})$, $i=0,1,2$, $p\in[1,\infty[$, the vector space of all processes ϕ satisfying conditions (a), (b_i) and

$$(\tilde{c}) \qquad \int_{R_{z_0}} |\phi_z|^p \,\mathrm{d}\Lambda_z < \infty \ \text{ a.s.}$$

and by $\tilde{\mathscr{H}}^p(\mathscr{F})$ the vector space of all processes Ψ satisfying conditions (d), (e), (f) and

$$(\tilde{g}) \qquad \int_{R^2_{z_0}} |\Psi_{xy}|^p \,\mathrm{d}\Lambda_x \,\mathrm{d}\Lambda_y < \infty \ \text{ a.s.}$$

Let $W=(W_z, z\in R_{z_0})$ be a real $\mathscr{F}$-Brownian martingale (resp. $\mathscr{F}$-i-Brownian sheet, $i=1,2$) and let $\phi\in\mathscr{H}_0^2(\mathscr{F})$ (resp. $\mathscr{H}_i^2(\mathscr{F})$, $i=1,2$). Then the stochastic integral

$$(\phi\cdot W)_z = \int_{R_z} \phi_x \,\mathrm{d}W_x$$

is defined as in [2] and [19] (here $\mathscr{F}$ is not supposed to have property F4); the process $\phi\cdot W$ is a square-integrable martingale (resp. i-martingale) such that

$$\mathbf{E}(\phi\cdot W)_z^2 = \int_{R_z} \mathbf{E}(\phi_x^2)\,\mathrm{d}\Lambda_x$$

and $\phi\cdot W$ has a continuous version (resp. measurable version continuous in s if $i=1$ and in t if $i=2$) with which we identify it.

For $\Psi\in\mathscr{H}^2(\mathscr{F})$ and for an $\mathscr{F}$-Brownian sheet W with $\mathscr{F}$ having property F4, the double integral and the mixed integrals

$$(\Psi\cdot WW)_z = \int_{R_z^2} \Psi_{xy}\,\mathrm{d}W_x\,\mathrm{d}W_y,$$

$$(\Psi\cdot W\Lambda)_z = \int_{R_z^2} \Psi_{xy}\,\mathrm{d}W_x\,\mathrm{d}\Lambda_y, \qquad (\Psi\cdot \Lambda W)_z = \int_{R_z^2} \Psi_{xy}\,\mathrm{d}\Lambda_x\,\mathrm{d}W_y \tag{1.1}$$

define respectively a continuous square integrable martingale, 1-martingale with bounded variations in t, 2-martingale with bounded variations in s and

$$\mathbf{E}(\Psi\cdot WW)_z^2 = \int_{R_z^2} \mathbf{E}(\Psi_{xy}^2)\,\mathrm{d}\Lambda_x\,\mathrm{d}\Lambda_y. \tag{1.2}$$

Given an $\mathbb{R}^2$-valued $\mathscr{F}$-Brownian sheet $W=(W^1, W^2)$, one can similarly define the mixed stochastic integral $\Psi\cdot W^1W^2$ which is a continuous square integrable martingale such that

$$\mathbf{E}(\Psi\cdot W^1W^2)_z^2 = \int_{R_z^2} \mathbf{E}(\Psi_{xy}^2)\,\mathrm{d}\Lambda_x\,\mathrm{d}\Lambda_y. \tag{1.3}$$

According to [3, 21] the above stochastic integrals can be extended to elements of $\tilde{\mathscr{H}}_i^2(\mathscr{F})$ and $\tilde{\mathscr{H}}^2(\mathscr{F})$ by a sort of localization.

Finally, we mention that, for $\phi \in \tilde{\mathscr{H}}_i^1(\mathscr{F})$, $i = 0, 1, 2$, the Stieljes integral

$$(\phi \cdot \Lambda)_z = \int_{R_z} \phi_x \, \mathrm{d}\Lambda_x \tag{1.4}$$

defines an $\mathscr{F}^i$-adapted process with bounded variation.

1.2.6

A continuous process Z is called a semimartingale (resp. a p-integrable semimartingale, $p=2m$, $m \in N^*$) of an R^n-valued $\mathscr{F}$-Brownian sheet $W = (W^1, \ldots, W^n)$ if there exist processes θ, ϕ^i in $\tilde{\mathscr{H}}_0^2(\mathscr{F})$ (resp. $\mathscr{H}_0^p(\mathscr{F})$) for $i = 1, \ldots, n$, and processes Ψ^{ij}, f^i, g^i in $\tilde{\mathscr{H}}^2(\mathscr{F})$ (resp. $\mathscr{H}^p(\mathscr{F})$) for $i, j = 1, \ldots, n$ such that

$$Z = \theta \cdot \Lambda + \sum_{i=1}^{n} \phi^i \cdot W^i + \sum_{i,j=1}^{n} \Psi^{ij} \cdot W^i W^j + \sum_{i=1}^{n} f^i \cdot \Lambda W^i + \sum_{i=1}^{n} g^i \cdot W^i \Lambda. \tag{1.5}$$

By partially integrating formula (1.5) one can write

$$Z_{st} = \sum_{i=1}^{n} \int_{R_{st}} U^i_{t;uv} \, \mathrm{d}W^i_{uv} + \int_{R_{st}} V_{t;uv} \, \mathrm{d}\Lambda_{uv} \tag{1.6}$$

where U^i, V are measurable on $[0, t_0] \times R_{z_0}$ and, for each t in $]0, t_0]$, U^i_t, V_t are adapted to the filtration $(\mathscr{F}_{ut}, u \in [0, s_0])$, vanish for $v > t$ and belong to $\tilde{\mathscr{H}}_1^2(\mathscr{F})$ (resp. $\mathscr{H}_1^p(\mathscr{F})$). More generally, such a process Z is called a 1-semimartingale (resp. p-integrable 1-semimartingale) of W. Similarly, Z can be represented as a 2-semimartingale (resp. p-integrable 2-semimartingale).

The following Girsanov type theorem extends those given in [23, 4, 7]. We recall that the filtration $\mathscr{F}$ is not supposed to have property F4.

Theorem 1.2. *Let $Y = (Y_z, z \in R_{z_0})$ be an $\mathbb{R}^n$-valued $\mathscr{F}$-Brownian martingale and let H be an $\mathbb{R}^n$-valued process whose components are in $\mathscr{H}_0^2(\mathscr{F})$.*

(a) *Then the process $L = (L_z, z \in R_{z_0})$ defined by*

$$L_z = \exp\left\{ \sum_{i=1}^{n} \int_{R_z} H_x^k \, \mathrm{d}Y_x - \tfrac{1}{2} \sum_{i=1}^{n} \int_{R_z} (H_x^k)^2 \, \mathrm{d}\Lambda_x \right\} \tag{1.7}$$

is a 1- and/or 2-martingale iff

$$\mathbf{E}(L_{z_0}) = 1. \tag{1.8}$$

(b) *Under this condition the probability $\mathbf{Q}$, defined on $\mathscr{F}_{z_0}$ by*

$$\mathbf{Q}(A) = E_p(L_{z_0} I(A)), \quad A \in \mathscr{F}_{z_0} \tag{1.9}$$

is equivalent to $\mathbf{P}$, *and the process*

$$W = Y - H \cdot A \tag{1.10}$$

is an $(\mathcal{F}, \mathbf{Q})$*-Brownian martingale.*

Proof. The above assertions are immediate consequences of the one-parameter Girsanov theorem [6]. We only show that, under condition (1.8), W is an $(\mathcal{F}, \mathbf{Q})$-Brownian martingale. In fact, for any choice of points $0 \le t_1 < t_2 < \cdots < t_{p+1} = t_0$, the process

$$\tilde{W} = ((W^k_{st_{j+1}} - W^k_{st_j})(t_{j+1} - t_j)^{-1/2},\ k = 1, \ldots, n, j = 1, \ldots, p;\ s \in [0, s_0])$$

is an $n \times p$-dimensional $(\mathcal{F}^1, \mathbf{Q})$-Brownian motion. Therefore, according to Proposition 1.1, W is an $\mathcal{F}$-1-Brownian sheet. By a symmetric argument, W can be shown to be an $\mathcal{F}$-2-Brownian sheet; hence W is an $\mathcal{F}$-Brownian martingale.

We shall need to evaluate stochastic integrals of Y under the probability $\mathbf{Q}$ of Theorem 1.2. The following proposition provides a tool for that purpose. We suppose here that H and W are one-dimensional.

Proposition 1.3. *Assume that condition* (1.8) *of Proposition* 1.2 *is satisfied and suppose that* $\mathcal{F}$ *verifies* F4 *under* $\mathbf{P}$. *Then, for* $\phi \in \mathcal{H}_i(\mathcal{F}, \mathbf{Q})$, $i = 0, 1, 2$, *the r.v.*

$$I(\phi) = \int_{R_{z_0}} \phi_z \, dW_z + \int_{R_{z_0}} \phi_z H_z \, dA_z$$

where the stochastic integral is evaluated under $\mathbf{Q}$, *coincides a.s. with the stochastic integral*

$$J(\phi) = \int_{R_{z_0}} \phi_z \, dY_z$$

evaluated under $\mathbf{P}$.

Proof. The assertion trivially holds if ϕ is an elementary process as the one used in the construction of stochastic integrals [2]. For an arbitrary $\phi \in \mathcal{H}_i^2(\mathcal{F}, \mathbf{Q})$, let $(\phi_n, n \ge 1)$ be a sequence of such elementary processes converging to ϕ in $\mathcal{H}_i^2(\mathcal{F}, \mathbf{Q})$. Then the sequence $(I(\phi_n), n \ge 1)$ converges to $I(\phi)$ in the quadratic mean under $\mathbf{Q}$; hence in probability under both $\mathbf{Q}$ and $\mathbf{P}$. But according to [22, Lemma 1, p. 771], the sequence $(J(\phi_n), n \ge 1)$ converges to $J(\phi)$ in probability. Therefore $I(\phi) = J(\phi)$ a.s.

2. Model and method

2.1. The model

The filtering equations will be established for the following signal and observation model:

- The signal to be filtered is a semimartingale X of a Brownian sheet B as defined in Section 1.2.6.
- The observation process Y verifies

$$Y = H \cdot \Lambda + W \tag{2.1}$$

where $H \in \mathscr{H}_0^2(\mathscr{F}^B)$ (usually H is a causal functional of only the signal X) and W (the observation noise) is a Brownian sheet independent of B.

The model will be constructed by the reference probability method which was first proposed in [25] and shown later in [1, 13, 15, 16] to be a powerful method in establishing filtering equations.

Let B and Y be two independent real Brownian sheets indexed by R_{z_0}, given on their canonical spaces $(\Omega^B, \mathscr{A}^B, \mathbf{P}^B)$ and $(\Omega^Y, \mathscr{A}^Y, \mathbf{P}^Y)$, respectively. The reference probability space, denoted by $(\Omega, \mathscr{A}, \mathbf{P})$, is defined as the completion of the product probability space $(\Omega^B \times \Omega^Y, \mathscr{A}^B \otimes \mathscr{A}^Y, \mathbf{P}^B \otimes \mathbf{P}^Y)$. The two independent Brownian sheets B and Y can be considered as defined on this space. We then denote by $\mathscr{F}$, $\mathscr{B}$ and $\mathscr{G}$ the natural filtrations, on $(\Omega, \mathscr{A}, \mathbf{P})$, of (B, Y), B and Y respectively, and we have $\mathscr{A} = \mathscr{F}_{z_0}$. We remark that filtrations $\mathscr{F}$, $\mathscr{B}$ and $\mathscr{G}$ possess property F4 under $\mathbf{P}$.

From now onwards, we shall suppose that the following hypotheses are verified on $(\Omega, \mathscr{A}, \mathbf{P})$.

H1: X is a semimartingale of B having the representation

$$X = \theta \cdot \Lambda + \Phi \cdot B + f \cdot \Lambda B + g \cdot B\Lambda + \Psi \cdot BB \tag{2.2}$$

where $\theta, \Phi \in \mathscr{H}_0^2(\mathscr{B})$ and $f, g, \Psi \in \mathscr{H}^2(\mathscr{B})$.

H2: $H \in \mathscr{H}_0^2(\mathscr{B})$ and the process L defined by

$$L = \exp\{H \cdot Y - \tfrac{1}{2}H^2 \cdot \Lambda\} \tag{2.3}$$

is an $(\mathscr{F}, \mathbf{P})$-martingale on R_{z_0}.

According to Theorem 1.2, hypothesis H2 implies that the probability $\mathbf{Q}$ defined on $\mathscr{A}$ as that of (1.9) is equivalent to $\mathbf{P}$. Moreover, under $\mathbf{Q}$, the process (B, W) is an $\mathbb{R}^2$-valued $\mathscr{F}$-Brownian martingale where

$$W = Y - H \cdot \Lambda. \tag{2.4}$$

We see that $\mathbf{P}$ and $\mathbf{Q}$ coincide on $\mathscr{B}_{z_0}$. Thus X is a semimartingale of the Brownian sheet B under both $\mathbf{P}$ and $\mathbf{Q}$, and H is an element of $\mathscr{H}_0^2(\mathscr{B}, \mathbf{P})$ or $\mathscr{H}_0^2(\mathscr{B}, \mathbf{Q})$ indistinctly. Under probability $\mathbf{Q}$, equation (2.4) is the exact replica of the observation equation (2.1). Consequently, under hypotheses H1 and H2, the 'signal and observation model' described above is well defined on $(\Omega, \mathscr{A}, \mathbf{Q})$. The advantage in starting from the reference probability space $(\Omega, \mathscr{A}, \mathbf{P})$ lies in the fact that this space is a product probability space that simplifies the elaboration of projection theorems and in the fact that filtrations $\mathscr{F}$, $\mathscr{B}$ and $\mathscr{G}$ have property F4 allowing us to fully apply the theory of stochastic calculus. We shall carry out all the computations on $(\Omega, \mathscr{A}, \mathbf{P})$ with respect to filtration $\mathscr{F}$ or $\mathscr{G}$, but the stochastic integrals appearing in the filtering equations may be equivalently computed with respect to $\mathscr{F}$ and $\mathbf{Q}$ as a consequence of the following result.

Proposition 2.1. *Under probability* $\mathbf{Q}$, $\mathscr{F}^{B,W}$ *coincides with* $\mathscr{F}$ *and, therefore,* (B, W) *is an* $\mathscr{F}$*-Brownian sheet.*

Proof. The proof is an immediate consequence of (2.4).

Finally, we mention that, under probability $\mathbf{Q}$, filtration $\mathscr{G}$ no longer possesses property F4, as can be deduced from a result of [7]. This is one of the reasons for our choice of the reference probability method.

2.2. Projection of processes

As in the one-parameter case, filtering equations for X will be expressed in terms of causal estimates, with respect to Y, of processes depending on the model. We therefore have to characterize $(\mathscr{G}, \mathbf{Q})$-projections of processes. For an $\mathscr{F}$-adapted and $\mathbf{Q}$-integrable process Z, a $(\mathscr{G}, \mathbf{Q})$-projection is a process $Z(\cdot/\cdot)$ such that

$$\forall y, \forall z, \quad Z(y/z) = E_{\mathbf{Q}}(Z_y/\mathscr{G}_z) \text{ a.s.}$$

It is of course important that the process $Z(\cdot/\cdot)$ has good trajectorial properties. But, by the Bayes formula, the above condition on $Z(\cdot/\cdot)$ can also be written as follows:

$$\forall y, \forall z, \quad Z(y/z) = E_{\mathbf{P}}(L_{y \vee z} Z_y/\mathscr{G}_z)/E_{\mathbf{P}}(L_z/\mathscr{G}_z) \text{ a.s.} \tag{2.5}$$

We see that the characterization of $(\mathscr{G}, \mathbf{Q})$-projections can be brought to that of $(\mathscr{G}, \mathbf{P})$-projections that we define below.

To any $\mathbf{P}$-integrable r.v. V, we associate an r.v. $k(V)$, $\mathbf{P}$-a.e. defined by the formula

$$k(V)(\omega^B, \omega^Y) = k(V)(\omega^Y) = \int_{\Omega^B} V(\omega, \omega^Y) P^B(\mathrm{d}\omega). \tag{2.6}$$

Obviously, $k(V)$ is a version of $E_{\mathbf{P}}(V/\mathscr{G}_{z_0})$ and if V is $\mathscr{F}_z$-measurable, then $k(V)$ is a version of $E_{\mathbf{P}}(V/\mathscr{G}_z)$. Or equivalently, for all z, $\mathscr{F}_z$ and $\mathscr{F}_{z_0}$ are conditionally independent given $\mathscr{G}_z$. This property was shown, in [1] and [15], to be a fundamental tool in various extensions of the reference probability method.

For an arbitrary $\mathbf{P}$-integrable process Z, we put $kZ = (k(Z_z), z \in R_{z_0})$. It is worth noticing that if Z is $\mathscr{F}$-adapted (resp. $\mathscr{F}^i$-adapted, $i = 1, 2$), then kZ is $\mathscr{G}$-adapted (resp. $\mathscr{G}^i$-adapted, $i = 1, 2$) and, for all z, kZ_z is a version of $E_{\mathbf{P}}(Z_z/\mathscr{G}_z)$ (resp. $E_{\mathbf{P}}(Z_z/\mathscr{G}^i_z), i = 1, 2$). Now, we would like to define a modification KZ of kZ with desired trajectorial properties. In fact, KZ will only be defined in the following three cases.

Case (i) For a process Z in $H^p(\mathscr{F}, \mathbf{P})$, $(p \geq 1)$, kZ is defined everywhere on $\Omega \times R_{z_0}$ except on an evanescent set and we put $KZ = kZ$. We note that $KZ \in H^p(\mathscr{G}, \mathbf{P})$. The projector K so defined on $H^p(\mathscr{F}, \mathbf{P})$ preserves the limit properties of the trajectories. In particular, if Z is continuous, then so is KZ.

Case (ii) If kZ has any continuous modification, this modification will be denoted by KZ.

Case (iii) If Z is a process in $\mathscr{H}_0^p(\mathscr{F})$ (resp. $\mathscr{H}_i^p(\mathscr{F}), i = 1, 2$), for some $p \geq 1$, then kZ is defined $\Lambda \otimes \mathbf{P}$-a.e. on $R_{z_0} \times \Omega$ and, for almost all $z \in R_{z_0}$, $(kZ)_z$ is a version of $E_{\mathbf{P}}(Z_z/\mathscr{G}_z)$ (resp. $E_{\mathbf{P}}(Z_z/\mathscr{G}_z^i), i = 1, 2$). We define KZ as any process in the equivalence class of kZ. Finally if $Z \in \mathscr{H}^p(\mathscr{F})$, we define KZ in the same way. We only note in this case that for almost all $(x, y) \in R_{z_0}^2$, $(KZ)_{x,y}$ is a version of $E_{\mathbf{P}}(Z_{x,y}/\mathscr{G}_{x \vee y})$.

Case (i) applies to continuous square-integrable martingales (resp. square-integrable proper i-martingales of [20], $i = 1, 2$). According to Cairoli–Doob's [2] (resp. Wong–Zakai's [20]) maximal inequalities, such a process, say Z, belongs to $H^1(\mathscr{F}, \mathbf{P})$ and KZ is a continuous square-integrable $\mathscr{G}$-martingale (resp. proper $\mathscr{G}$-i-martingale, $i = 1, 2$).

Case (ii) applies to the martingale L of (2.3) as shown by the following proposition.

Proposition 2.2. *The martingale L of* (2.3) *is $\boldsymbol{L} \log^+ \boldsymbol{L}$-bounded and, for almost all ω, $(KL)(\omega)$ is bounded below by a positive constant (depending on ω).*

Proof. We have

$$|\log^+ L_z| \leq |\log L_z| = \left| \int_{R_z} H_x \, \mathrm{d}Y_x - \tfrac{1}{2} \int_{R_z} H_x^2 \, \mathrm{d}\Lambda_x \right|,$$

$$|\log^+ L_z| \leq \left| \int_{R_z} H_x \, \mathrm{d}W_x + \tfrac{1}{2} \int_{R_z} H_x^2 \, \mathrm{d}\Lambda_x \right|.$$

Therefore,

$$E_{\mathbf{P}}(L_z \log^+ L_z) = E_{\mathbf{Q}}(\log^+ L_z) \leq E_{\mathbf{Q}} \left| \int_{R_z} H_x \, \mathrm{d}W_x \right| + \tfrac{1}{2} E_{\mathbf{Q}} \int_{R_z} H_x^2 \, \mathrm{d}\Lambda_x,$$

$$E_{\mathbf{Q}}(\log^+ L_z) \leq E_{\mathbf{Q}} \left(\int_{R_{z_0}} H_x^2 \, \mathrm{d}\Lambda_z \right)^{1/2} + \tfrac{1}{2} E_{\mathbf{Q}} \int_{R_z} H_x^2 \, \mathrm{d}\Lambda_x < \infty.$$

The second part of the proposition is deduced from [3, Lemma 2.5] after having noticed that $kL_{z_0} > 0$ a.s.

We can now express the projection theorem on square-integrable semimartingales of the Brownian sheet (B, Y), generalizing the one proved in [1] for one-parameter stochastic integrals.

Proposition 2.3. (a) *Let Z be the square-integrable semimartingale of the Brownian sheet* (Y, B) *defined by*

$$Z = \theta \cdot \Lambda + \phi \cdot Y + \phi' \cdot B + \Psi \cdot YY + \Psi' \cdot BB + \Psi'' \cdot BY$$
$$+ f \cdot \Lambda Y + f' \cdot \Lambda B + g \cdot Y\Lambda + g' \cdot B\Lambda \tag{2.7}$$

where $\theta \in \mathcal{H}_0^1(\mathcal{F})$; ϕ, $\phi' \in \mathcal{H}_0^2(\mathcal{F})$; Ψ, Ψ', Ψ'', f, f', g, $g' \in \mathcal{H}^2(\mathcal{F})$; *then* $K(Z)$ *is the square-integrable semimartingale of Y given by*

$$K(Z) = K(\theta) \cdot \Lambda + K(\phi) \cdot Y + K(\Psi) \cdot YY + K(f) \cdot \Lambda Y + K(g) \cdot Y\Lambda. \tag{2.8}$$

(b) *Let Z be the square-integrable* 1*-semimartingale of the Brownian sheet* (Y, B) *defined by*

$$Z_{st} = (V_{t,\cdot} \cdot \Lambda)_{st} + (U_{t,\cdot} \cdot Y)_{st} + (U'_{t,\cdot} \cdot B)_{st}, \quad (s, t) \in R_{z_0} \tag{2.9}$$

where V, U, U' satisfy conditions listed in Section 1.2.6; *then* $K(Z)$ *is the square-integrable* 1*-semimartingale of Y given by*

$$K(Z)_{st} = (K(V_{t,\cdot}) \cdot \Lambda)_{st} + (K(U_{t,\cdot}) \cdot V)_{st}. \tag{2.10}$$

(c) *An analogous assertion holds for a square-integrable* 2*-semimartingale.*

Proof. We compute the projection of (2.7) and (2.9) term by term by the same method as in [1] and [15]. According to maximal inequalities in [2] and [20], the terms of the right-hand sides of (2.7) and (2.9) define processes belonging to $H^2(\mathcal{F}, \mathbf{P})$. Therefore, their $(\mathcal{G}, \mathbf{P})$-projection (as well as that of Z) are well defined by the projection operator K of Case (i) considered above. On the other hand, when the integrands of these terms are appropriate elementary processes, one can see that (2.8) and (2.10) hold. The final conclusion is reached by using the density of elementary processes in the corresponding spaces $\mathcal{H}^2(\mathcal{F}, \mathbf{P})$ and $\mathcal{H}_i^2(\mathcal{F}, \mathbf{P})$.

To end with process projections, we now define $(\mathcal{F}, \mathbf{Q})$-projections of processes, compatible with the above definitions of $(\mathcal{G}, \mathbf{P})$-projections.

For any $\mathcal{F}_y$-adapted and $\mathbf{Q}$-integrable r.v. Z_y, we put

$$(Z_y/z) = k(L_{y \vee z} Z_y)/(KL)_z \tag{2.11}$$

By Proposition 2.2 this ratio is defined a.e. and is a version of $E_{\mathbf{Q}}(Z_y/\mathcal{G}_z)$.

If Z is an $\mathcal{F}$-adapted and $\mathbf{Q}$-integrable process, then the process defined by (Z_y/z) (and indexed by $R_{z_0}^2$ or by R_{z_0} when either y or z is fixed) may have a modification with interesting trajectorial properties according to those of the process defined by $k(L_{y \vee z} Z_y)$. In particular, whenever LZ enters one of Cases (i), (ii) or (iii) where $K(LZ)$ is defined as above, then we define the process (Z_z/z) by the ratio $K(LZ)/KL$. But these are not the only interesting modifications of (Z_z/z). In fact, if $k(LZ)$ happens to have a representation as semimartingale of Y, then obviously (Z_z/z) has a continuous modification. Consequently, this kind of 'good' modifications to be defined for the process (Z_y/z) or (Z_z/z) will appear clearly in the context. That is why we do not insist on classifying them.

2.3. Unnormalized filtering equations

Let X be the square-integrable semimartingale defined by (2.2). Then it has the following representations as 1-semimartingale:

$$X_{st} = \int_{R_{st}} U_{t,uv}\, dB_{uv} + \int_{R_{st}} V_{t,uv}\, dA_{uv} \tag{2.12}$$

where

$$U_{t,uv} = \phi_{uv} + \int_{R_{st}} f_{ab,uv}\, dA_{ab} + \int_{R_{st}} \Psi_{ab,uv}\, dB_{ab}, \tag{2.13}$$

$$V_{t,uv} = \theta_{uv} + \int_{R_{st}} g_{ab,uv}\, dB_{ab}, \tag{2.14}$$

and, as 2-semimartingale,

$$X_{st} = \int_{R_{st}} \tilde{U}_{s,ab}\, dB_{ab} + \int_{R_{st}} \tilde{V}_{s,ab}\, dA_{ab} \tag{2.15}$$

where

$$\tilde{U}_{s,ab} = \phi_{ab} + \int_{R_{st}} g_{ab,uv}\, dA_{uv} + \int_{R_{st}} \Psi_{ab,uv}\, dB_{uv}, \tag{2.16}$$

$$\tilde{V}_{s,ab} = \theta_{ab} + \int_{R_{st}} f_{ab,uv}\, dB_{uv}. \tag{2.17}$$

In Section 4 we shall give conditions under which the Ito differentiation rules of [24] applies to the product LX, providing the following representations as a semimartingale of the Brownian sheet (B, Y):

$$(LX)_{st} = \int_{R_{st}} L_{ut}\{X_{ut}H_{uv}\, dY_{uv} + U_{t,uv}\, dB_{uv} + V_{t,uv}\, dA_{uv}\}, \tag{2.18}$$

$$(LX)_{st} = \int_{R_{st}} L_{sb}\{X_{sb}H_{ab}\, dY_{ab} + \tilde{U}_{s,ab}\, dB_{ab} + V_{s,ab}\, dA_{ab}\}, \tag{2.19}$$

$$(LX)_{st} = \int_{R_{st}} L_{uv}\{\theta_{uv}\, dA_{uv} + X_{uv}H_{uv}\, dY_{uv} + \phi_{uv}\, dB_{uv}\}$$

$$+ \int_{R_{st}^2} I_{uv,ab}L_{ub}\{V_{b,uv}H_{ab}\, dY_{ab}\, dA_{uv} + \tilde{V}_{u,ab}H_{uv}\, dA_{ab}\, dY_{uv}$$

$$+X_{ub}H_{uv}H_{ab}\,\mathrm{d}Y_{ab}\,\mathrm{d}Y_{uv}\}$$

$$+\int_{R_{st}^2} I_{uv,ab}L_{ub}\{f_{ab,uv}\,\mathrm{d}A_{ab}\,\mathrm{d}B_{uv}+g_{ab,uv}\mathrm{d}B_{ab}\,\mathrm{d}A_{uv}$$

$$+\Psi_{ab,uv}\,\mathrm{d}B_{ab}\,\mathrm{d}B_{uv}$$

$$+U_{b,uv}H_{ab}\,\mathrm{d}Y_{ab}\,\mathrm{d}B_{uv}+\tilde{U}_{u,ab}H_{uv}\,\mathrm{d}B_{ab}\,\mathrm{d}Y_{uv}\}. \tag{2.20}$$

In case LX is a square-integrable semimartingale, Proposition 2.3 applies and we get the representation of $K(LX)$ as a square-integrable semimartingale of Y.

Proposition 2.4. *If LX is a square-integrable semimartingale of (B, Y), then $K(LX)$ has the following representations*:

$$K(LX)_{st}=\int_{R_{st}} K(L)_{ut}(X_{ut}H_{uv}/ut)\,\mathrm{d}Y_{uv}+\int_{R_{st}} K(L)_{ut}(V_{t,uv}/ut)\,\mathrm{d}A_{uv}, \tag{2.21}$$

$$K(LX)_{st}=\int_{R_{st}} K(L)_{sb}(X_{sb}H_{ab}/sb)\,\mathrm{d}Y_{ab}+\int_{R_{st}} K(L)_{sb}(\tilde{V}_{s,ab}/sb)\,\mathrm{d}A_{ab}, \tag{2.22}$$

$$K(LX)_{st}=\int_{R_{st}} K(L)_{uv}\{(\theta_{uv}/uv)\,\mathrm{d}A_{uv}+(X_{uv}H_{uv}/uv)\,\mathrm{d}Y_{uv}\}$$

$$+\int_{R_z^2} I_{uv,ab}K(L)_{ub}\{(V_{b,uv}H_{ab}/ub)\,\mathrm{d}Y_{ab}\,\mathrm{d}A_{uv}$$

$$+(\tilde{V}_{u,ab}H_{uv}/ub)\,\mathrm{d}A_{ab}\,\mathrm{d}Y_{uv}+(X_{ub}H_{ab}H_{uv}/ub)\,\mathrm{d}Y_{ab}\,\mathrm{d}Y_{uv}\}. \tag{2.23}$$

Proof. The proof is an immediate consequence of the projection proposition 2.3.

Remark. When L is a square-integrable martingale, we get the representation of $K(L)$ in terms of Y, similar to Doleans–Dade type equations, already given in [23] for a bounded H. These equations can be deduced from (2.21), (2.22) and (2.23) by setting $X=1$ and θ, ϕ, V, $\tilde{V}=0$, and by adding 1 to their right-hand sides.

In Section 4, we shall see that (2.21), (2.22) and (2.23) can be written under more general conditions. Whenever these equations are valid, we call them horizontal, vertical and diagonal unnormalized filtering equations, respectively.

In order to obtain filtering equations expressing (X_z/z) as semimartingales of Y, we have to apply the adequate Ito differentiation rules to the ratio $K(LX)/K(L)$. In Section 3 we shall obtain them by formal computations and give, in Section 4, various sets of hypotheses under which the obtained results are valid.

3. Filtering equations

Following the formulation of the two-parameter causal recursive filtering problems as in [17] and [18], three types of filtering equations are obtained for the 'causal estimates' $(X_{\sigma\tau}/st)$ of $X_{\sigma\tau}$, when (σ,τ) runs over the 'presents' of R_{st}, corresponding to various displacements of the point (s,t) towards the 'futures'. For a fixed height t (resp. width s), the equation expressing the stochastic differential in s (resp. in t) of $(X_{\sigma\tau}/st)$ for $(\sigma,\tau)\in\partial^1 R_{st}$ (resp. $\partial^2 R_{st}$) is called the *horizontal* (resp. *vertical*) filtering equation. Similarly, the equation expressing the stochastic differential in (s,t) of $(X_{\sigma\tau}/st)$ for $(\sigma,\tau)\in\partial R_{st}$ is called the *diagonal* filtering equation.

Although theorems of representation by innovations are not used here (cf. [5] for the one-parameter case and [8, 16] for the two-parameter Gaussian case), filtering equations will also be expressed in terms of various types of innovation processes that we first define below.

Definition 3.1. The process $\nu^0=(\nu^0_z, z\in R_{z_0})$ defined by

$$\nu^0_z = Y_z - \int_{R_z} (H_x/x)\,\mathrm{d}A_x \tag{3.1}$$

is called the *corner innovation*.

For a given t' in $[0,t_0]$, the process $\nu^{1,t'}=(\nu^{1,t'}_z, z\in R_{s_0t'})$ defined by

$$\nu^{1,t'}_z = Y_z - \int_{R_z} (H_{uv}/u,t')\,\mathrm{d}A_{uv} \tag{3.2}$$

is called the *horizontal innovation* of height t'.

For a given s' in $[0,s_0]$, the process $\nu^{2,s'}=(\nu^{2,s'}_z, z\in R_{s't_0})$ defined by

$$\nu^{2,s'}_z = Y_z - \int_{R_z} (H_{ab}/s'b)\,\mathrm{d}A_{ab} \tag{3.3}$$

is called the *vertical innovation* of width s'.

The next proposition which extends the results given in [23] justifies that we call innovations the above processes ν.

Proposition 3.2. *Under probability* $\mathbf{Q}$ *the following assertions hold*:

(a) ν^0 *is a weak* $\mathcal{G}$*-martingale.*

(b) *For a fixed* t' (*resp.* s'), *the process* $\nu^{1,t'}$ (*resp.* $\nu^{2,s'}$) *is a* 1*-Brownian sheet* (2*-Brownian sheet*) *with respect to the filtration* $(\mathcal{G}_z,\ z<(s_0,t'))$ (*resp.* $\mathcal{G}_z,\ z<(s',t_0)$)).

Proof. (a) is proved as in [23].

We prove the assertion (b) for the horizontal innovation. According to its definition, $\nu^{1,t'}$ is a continuous process on $R_{s_0t'}$. Formula (2.21), written for L, gives

$$K(L)_{st'} = \exp\left\{\int_{R_{st'}} [(H_{uv}/ut')\,\mathrm{d}Y_{uv} - \tfrac{1}{2}(H_{uv}/ut')^2\,\mathrm{d}A_{uv}]\right\}.$$

Let $\mathbf{P}'$ and $\mathbf{Q}'$ be the restrictions to $\mathcal{G}_{s_0t'}$ of $\mathbf{P}$ and $\mathbf{Q}$. $(KL)_{s_0t'}$ is the Radon–Nikodym derivative of $\mathbf{Q}'$ with respect to $\mathbf{P}'$. Then an application of the one-parameter Girsanov theorem shows that the process $\nu^{1,t'}$ is a 1-Brownian sheet under the probability $\mathbf{Q}'$ (hence under $\mathbf{Q}$) with respect to $(\mathcal{G}_z, z < (s_0, t'))$.

In order to obtain the filtering equations, we make in this section, besides hypotheses H1 and H2, the following general hypothesis on the model:

H *Processes θ, ϕ, Ψ, f, g and H are such that the unnormalized filtering equations* (2.21), (2.22), (2.23) *are valid, the Ito differentiation rules that we shall apply are justified and the resulting integrals are meaningful.*

Section 4 is devoted to the conditions of validity of this hypothesis. Some of these conditions allow a unified approach for the nonlinear case of [11] and the linear Gaussian case of [8] and [17].

To shorten the expression of the filtering equations we introduce the following notations. For r.v.'s Z_1, Z_2, Z_3 on $(\Omega, \mathcal{F}_{z_0}, \mathbf{Q})$ we put

$$\rho_z(Z_1, Z_2) = \left(\prod_{k=1}^{2} (Z_k - (Z_k/z))/z\right),$$
$$\rho_z(Z_1, Z_2, Z_3) = \left(\prod_{k=1}^{3} (Z_k - (Z_k/z))/z\right). \tag{3.4}$$

According to the definition of $(\mathcal{G}, \mathbf{Q})$-projections, these quantities coincide a.s. with the corresponding conditional moments.

3.1. Lateral filtering and smoothing equations

Lateral filtering equations were obtained in [11]; we give here directly the smoothing equations.

Theorem 3.3 (Lateral smoothing equations). *For $(\sigma, \tau) < (s, t)$, we have*

$$(X_{\sigma\tau}/st) = \int_{R_{\sigma\tau}} (V_{\tau,uv}/ut)\mathrm{d}A_{uv} + \int_{R_{\sigma\tau}} \rho_{ut}(X_{u\tau}, H_{uv})\,\mathrm{d}\nu^{1,t}_{uv} + \int_{]\sigma,s]\times]0,t]} \rho_{ut}(X_{\sigma\tau}, H_{uv})\,\mathrm{d}\nu^{1,t}_{uv}, \tag{3.5}$$

$$(X_{\sigma\tau}/st) = \int_{R_{\sigma\tau}} (\tilde{V}_{\sigma,uv}/sv)\,\mathrm{d}A_{uv} + \int_{R_{s\tau}} \rho_{sv}(X_{\sigma v}, H_{uv})\,\mathrm{d}\nu^{2,s}_{uv}$$
$$+ \int_{]0,s]\times]\tau,t]} \rho_{sv}(X_{\sigma\tau}, H_{uv})\,\mathrm{d}\nu^{2,s}_{uv}. \tag{3.6}$$

Proof. For $(\sigma, \tau) = (s, t)$, equations (3.5) (resp. (3.6)) are obtained by applying the one-parameter Ito differentiation rule to the quotient of $K(LX)$ by $K(L)$ both written as 1-semimartingales (resp. 2-semimartingales) deduced from (2.21) (resp. (2.22)). For $(\sigma, \tau) < (s, t)$ defining the smoothing case, (3.5) and (3.6) are obtained by writing the corresponding filtering equations for the process $X^{\sigma\tau}$ defined by (3.7):

$$X^{\sigma\tau}_{st} = X_{s\wedge\sigma, t\wedge\tau}. \tag{3.7}$$

Remark. Writing (3.5) for $s = \sigma$, we obtain, for all $\tau \leq t$,

$$(X_{s\tau}/st) = \int_{R_{s\tau}} (V_{\tau,uv}/u, t)\,\mathrm{d}A_{uv} + \int_{R_{st}} \rho_{ut}(X_{u\tau}, H_{uv})\,\mathrm{d}\nu^{1,t}_{uv}. \tag{3.8}$$

This set of equations can be considered as a recursive filtering equation as was pointed out in [18], for the one-parameter process $X_{s,\cdot}$, $s \in [0, s_0]$), where for almost all $\omega \in \Omega$, $X_{s,\cdot}(\omega)$ is a continuous real function on $[0, t_0]$. This point of view was adopted in [8] for the Gaussian case, where lateral filtering equations were deduced from the corresponding equations for Hilbert space valued processes. The analogy between equation (3.8) and the one-parameter filtering equation [5] is obvious. As a matter of fact, in the method used in deriving the above filtering equation, the two-parameter aspect of the model does not play a great role.

3.2. Diagonal filtering equation

The diagonal filtering equation expresses (X_z/z) as a $(\mathcal{G}, \mathbf{P})$-semimartingale of Y. It was obtained in [11] for a bounded H by applying the two-parameter Ito differentiation formula of [24] to the quotient of $K(LX)$ by $K(L)$. The computation of [11] was tremendously long. We propose here another method which only uses lateral filtering equations. This approach is similar to the one used in [24] for deriving the two-parameter Ito formula from the one-parameter formulas.

Theorem 3.4 (Diagonal filtering equation). *The estimate (X_{st}/st) satisfies the following equation*:

$$(X_{st}/st) = \int_{R_{st}} (\theta_{uv}/uv)\,\mathrm{d}A_{uv} + \int_{R_{st}} \rho_{uv}(X_{uv}, H_{uv})\{\mathrm{d}Y_{uv} - (H_{uv}/uv)\,\mathrm{d}A_{uv}\}$$
$$+ \int_{R_{st}} \left\{\int_v^t d_b \rho_{ub}(X_{ub}, H_{uv})\right\} \mathrm{d}Y_{uv}$$

$$-\int_{R_{st}}\left\{\int_v^t (H_{uv}/ub) d_b\rho_{ub}(X_{ub}, H_{uv})\right\} \mathrm{d}\Lambda_{uv}$$

$$+\int_{R_{st}}\left\{\int_a^s d_u\rho_{ub}(X_{ub}, H_{ab})\right\} \mathrm{d}Y_{ab}$$

$$-\int_{R_{st}}\left\{\int_a^s (H_{ab}/ub)\, d_u\rho_{ub}(X_{ub}, H_{ab})\right\} \mathrm{d}\Lambda_{ab}$$

$$-\int_{R_{st}^2} I_{ab,uv}\rho_{ub}(X_{ub}, H_{ab}, H_{uv})$$

$$\times\{\mathrm{d}Y_{ab}\,\mathrm{d}Y_{uv} - (H_{uv}/ub)\,\mathrm{d}Y_{ab}\,\mathrm{d}\Lambda_{uv}$$

$$-(H_{ab}/ub)\,\mathrm{d}\Lambda_{ab}\,\mathrm{d}Y_{uv} + (H_{ab}H_{uv}/ub)\,\mathrm{d}\Lambda_{ab}\,\mathrm{d}\Lambda_{uv}\} \quad (3.9)$$

where $d_b\rho_{ub}(X_{ub}, H_{uv})$ *(resp.* $d_u\rho_{ub}(X_{ub}, H_{ab})$*) represents the stochastic differential of* $\rho_{ub}(X_{ub}, H_{uv})$ *with respect to b (resp. u) given by the following formula* (3.10) *(resp.* (3.11)*):*

$$\rho_{ut}(X_{ut}, H_{uv}) = \rho_{uv}(X_{uv}, H_{uv}) + \int_{]0,u]\times]v,t]} \rho_{ub}(X_{ub}, H_{ab}, H_{uv})\,\mathrm{d}\nu_{ab}^{2,u}$$

$$+\int_{]0,u]\times]v,t]} \{\rho_{ub}(\tilde{V}_{u,ab}, H_{uv})$$

$$-\rho_{ub}(X_{ub}, H_{ab})\rho_{ub}(H_{ab}, H_{uv})\}\,\mathrm{d}\Lambda_{ab}, \quad (3.10)$$

$$\rho_{sb}(X_{sb}, H_{ab}) = \rho_{ab}(X_{ab}, H_{ab}) + \int_{]a,s]\times]0,b]} \rho_{ub}(X_{ub}, H_{ab}, H_{uv})\,\mathrm{d}\nu_{uv}^{1,b}$$

$$+\int_{]a,s]\times]0,b]} \{\rho_{ub}(V_{b,uv}, H_{ab})$$

$$-\rho_{ub}(X_{ub}, H_{uv})\rho_{ub}(H_{ab}, H_{uv})\}\,\mathrm{d}\Lambda_{uv}. \quad (3.11)$$

Proof. We start from the horizontal filtering equation

$$(X_{st}/st) = \int_{R_{st}} (V_{t,uv}/ut)\,\mathrm{d}\Lambda_{uv}$$

$$+\int_{R_{st}} \rho_{ut}(X_{ut}, H_{uv})\{\mathrm{d}Y_{uv} - (H_{uv}/ut)\,\mathrm{d}\Lambda_{uv}\} \quad (3.12)$$

and replace $(V_{t,uv}/ut)$, $\rho_{ut}(X_{ut}, H_{uv})$ and (H_{uv}/ut) by their vertical filtering equations.

The vertical filtering equation (3.6) gives

$$(X_{ut}/ut) = (X_{uv}/uv) + \int_{]0,u]\times]v,t]} (\tilde{V}_{u,ab}/ub)\,\mathrm{d}\Lambda_{ab}$$

$$+\int_{]0,u]\times]v,t]} \rho_{ub}(X_{ub}, H_{ab})\,\mathrm{d}\nu_{ab}^{2,u}. \quad (3.13)$$

Similarly, an adequate application of our method to $X_{ut}H_{uv}$, H_{uv} and $V_{t,uv}$ provides the following vertical filtering equations for $v \leq t$:

$$(X_{ut}H_{uv}/ut) = (X_{uv}H_{uv}/uv) + \int_{]0,u]\times]v,t]} (\tilde{V}_{u,ab}H_{uv}/ub)\,\mathrm{d}\Lambda_{ab}$$

$$+ \int_{]0,u]\times]v,t]} \rho_{ub}(X_{ub}H_{uv}, H_{ab})\,\mathrm{d}\nu^{2,u}_{ab}, \tag{3.14}$$

$$(H_{uv}/ut) = (H_{uv}/uv) + \int_{]0,u]\times]v,t]} \rho_{ub}(H_{uv}, H_{ab})\,\mathrm{d}\nu^{2,u}_{ab}, \tag{3.15}$$

$$(V_{t,uv}/ut) = (\theta_{uv}/uv) + \int_{]0,u]\times]v,t]} \rho_{ub}(V_{b,uv}, H_{ab})\,\mathrm{d}\nu^{2,u}_{ab}. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we obtain (3.10). Eq. (3.11) is obtained in the same way.

Eqs. (3.15) and (3.16) give

$$\rho_{ut}(X_{ut}, H_{uv})(H_{uv}/ut) =$$
$$= \rho_{uv}(X_{uv}, H_{uv})(H_{uv}/uv)$$
$$+ \int_{]0,u]\times]v,t]} \rho_{ub}(X_{ub}, H_{uv})\rho_{ub}(H_{ab}, H_{uv})\,\mathrm{d}\nu^{2,u}_{ab}$$
$$+ \int_v^t (H_{uv}/ub)\,\mathrm{d}_b\rho_{ub}(X_{ub}, H_{uv})$$
$$+ \int_{]0,u]\times]v,t]} \rho_{ub}(H_{ab}, H_{uv})\rho_{ub}(X_{ub}, H_{ab}, H_{uv})\,\mathrm{d}\Lambda_{ab} \tag{3.17}$$

where $d_b\rho_{ub}(X_{ub}, H_{uv})$ is the stochastic differential of $\rho_{ub}(X_{ub}, H_{uv})$ with respect to b.

By substituting (3.16), (3.10) and (3.17) in (3.12), we obtain

$$(X_{st}/st) = \int_{R_{st}} \{(\theta_{uv}/uv)\,\mathrm{d}\Lambda_{uv} + \rho_{uv}(X_{uv}, H_{uv})\,\mathrm{d}\nu^0_{uv}\}$$
$$\int_{R_{st}} \left\{\int_v^t \mathrm{d}_b\rho_{ub}(X_{ub}, H_{uv})\right\}\mathrm{d}Y_{uv}$$
$$- \int_{R_{st}} \left\{\int_v^t (H_{uv}/ub)\,\mathrm{d}_b\rho_{ub}(X_{ub}, H_{uv})\right\}\mathrm{d}\Lambda_{uv}$$
$$+ \int_{F_{st}} \left\{\int_{]0,u]\times]v,t]} [\rho_{ub}(V_{b,uv}, H_{uv})$$
$$- \rho_{ub}(X_{ub}, H_{uv})\rho_{ub}(H_{ab}, H_{uv})]\,\mathrm{d}\nu^{2,u}_{ab}\right\}\mathrm{d}\Lambda_{uv}$$
$$- \int_{R_{st}} \left\{\int_{]0,u]\times]v,t]} \rho_{ub}(H_{ab}, H_{uv})\rho_{ub}(X_{ub}, H_{ab}, H_{uv})\,\mathrm{d}\Lambda_{ab}\right\}\mathrm{d}\Lambda_{uv}. \tag{3.18}$$

Using definition (3.3) of ν^2, and (3.11) we finally get (3.9).

In fact, (3.9) is a clearer form of the equation given in [11], reproduced below.

Corollary 3.5. *Under the hypotheses of Theorem* 3.4, *equations* (3.9), (3.10) *and* (3.11) *give*

$$
\begin{aligned}
(X_{st}/st) = {} & \int_{R_{st}} \{(\theta_{uv}/u, v)\, \mathrm{d}A_{uv} + \rho_{uv}(X_{uv}, H_{uv})\, \mathrm{d}\nu^0_{uv}\} \\
& + \int_{R^2_{st}} I_{ab,uv}\{\rho_{ub}(\tilde{V}_{u,ab}, H_{uv}) \\
& \qquad - \rho_{ub}(X_{ub}, H_{ab})\rho_{ub}(H_{ab}, H_{uv})\}\, \mathrm{d}\nu^{1,b}_{uv}\, \mathrm{d}A_{ab} \\
& + \int_{R^2_{st}} I_{ab,uv}\{\rho_{ub}(V_{b,uv}, H_{ab}) \\
& \qquad - \rho_{ub}(X_{ub}, H_{uv})\rho_{ub}(H_{ab}, H_{uv})\}\, \mathrm{d}\nu^{2,u}_{ab}\, \mathrm{d}A_{uv} \\
& + \int_{R^2_{st}} I_{ab,uv}\rho_{ub}(X_{ub}, H_{ab}, H_{uv}) \\
& \qquad \times \{\mathrm{d}\nu^{2,u}_{ab}\, \mathrm{d}\nu^{1,b}_{uv} - \rho_{ub}(H_{ab}, H_{uv})\, \mathrm{d}A_{ab}\, \mathrm{d}A_{uv}\}.
\end{aligned}
\tag{3.19}
$$

In order to have the recursive diagonal filtering equation for X, it is sufficient to write (3.9) or (3.19) for the process X^ζ defined by (3.7) with $\zeta \in \partial R_{st}$. For instance, the equation of (X_ζ/st) is obtained from (3.9) by replacing X by X^ζ and

$$\int_{R_{st}} (\theta_{uv}/uv)\, \mathrm{d}A_{uv} \quad \text{by} \quad \int_{R_\zeta} (\theta_{uv}/uv)\, \mathrm{d}A_{uv}.$$

Remark. As outlined in Section 1.1, we do not have a satisfactory definition of multiple stochastic integrals with respect to the filtration $\mathscr{G}$ and probability **Q**. Nor do we have an intrinsic way of defining all the stochastic integrals with respect to the innovations. Nevertheless, (3.19) may alternatively be considered with respect to filtration $\mathscr{G}$ and probability **P** with stochastic integrals computed in Y and A or with respect to filtration $\mathscr{F}$ and probability **Q** with stochastic integrals computed in W (see (2.4)) and A. So far we do not know how to define intrinsic stochastic integrals with innovations.

3.3. Generalized Riccati equations

As in the one-parameter case, it would be interesting to have stochastic differential equations satisfied by the conditional covariance $\rho_{st}(X_{s_1t_1}, X_{s_2t_2})$ for $(s_1, t_1), (s_2, t_2) \in \partial R_{st}$. The method used in establishing (3.10) and (3.11) applies here. We then obtain the following equations written in differential form:

$$
\begin{aligned}
d_s\rho_{st}(X_{st_1}, X_{st_2}) = & \int_0^{t_1} \rho_{st}(X_{st_2}, V_{t_1,sv})\, \mathrm{d}\Lambda_{sv} \\
& + \int_0^{t_2} \rho_{st}(X_{st_1}, V_{t_2,sv})\, \mathrm{d}\Lambda_{sv} \\
& + \int_0^{t_1\wedge t_2} (U_{t_1,sv}U_{t_2,sv}/st)\, \mathrm{d}\Lambda_{sv} \\
& - \int_0^{t} \rho_{st}(X_{st_1}, H_{sv})\rho_{st}(X_{st_2}, H_{sv})\, \mathrm{d}\Lambda_{sv} \\
& + \int_0^{t} \rho_{st}(X_{st_1}, X_{st_2}, H_{sv})\, \mathrm{d}\nu_{sv}^{1,t}
\end{aligned}
\tag{3.20}
$$

for $t_1 \leq t$, $t_2 \leq t$,

$$
\begin{aligned}
d_s\rho_{st}(X_{s_1t}, X_{st_2}) = & \int_0^{t_2} \rho_{st}(X_{s_1t}, V_{t_2,sv})\, \mathrm{d}\Lambda_{sv} \\
& - \int_0^{t} \rho_{st}(X_{st_2}, H_{sv})\rho_{st}(X_{s_1t}, H_{sv})\, \mathrm{d}\Lambda_{sv} \\
& + \int_0^{t} \rho_{st}(X_{s_1t}, X_{st_2}, H_{sv})\, \mathrm{d}\nu_{sv}^{1,t}
\end{aligned}
\tag{3.21}
$$

for any $s_1 < s$ and $t_2 \leq t$.

Similar equations expressing $d_t\rho_{st}(X_{s_1t}, X_{s_2t})$ for $s_1 \leq s$, $s_2 \leq s$ and $d_t\rho_{st}(X_{s_1t}, X_{st_2})$ for $s_1 \leq s$ and $t_2 < t$ can be written in terms of $\nu^{2,s}$ by permuting the roles of s and t.

4. Conditions of validity of the filtering equations

In this section we express hypotheses on X and H justifying step by step all the computations made for the derivation of the filtering equations in the preceding section. Our motivation in the elaboration of these hypotheses is the formulation of a general approach applicable to both the nonlinear model of [11] and the Gaussian linear models of [8] and [17].

In order to avoid frequent repetitions we recall that everything is constructed on the reference probability space $(\Omega, \mathscr{A}, \mathbf{P})$ of Section 2.1, X is a square-integrable semimartingale of B given by (2.2); H is a $\mathscr{B}$-adapted process; L is defined by (2.3) whenever $H \cdot Y$ is well defined; and $\mathbf{Q}$ is the probability measure defined by $\mathrm{d}\mathbf{Q} = L_{z_0}\, \mathrm{d}\mathbf{P}$ when L is a martingale. We shall denote by $\mathbf{E}$ expectations under $\mathbf{P}$ and by $\mathbf{E}_{\mathbf{P}'}$ those under any other probability $\mathbf{P}'$. In all the situations considered here the validity of the model is guaranteed (cf. hypotheses H1 and H2 of Section 2.1).

The definition of a p-integrable semimartingale of a Brownian sheet was given in Section 1.2.6 and we often use the following majoration deduced in [7] from the maximal inequalities in [2] and [20].

Lemma 4.1. *If X is a $2n$-integrable semimartingale of a Brownian sheet with $n \in \mathbb{N}^*$, then we have*

$$\mathbf{E}[(X^*)^{2n}] < \infty, \tag{4.1}$$

$$\sup_{s,t} \left\{ \mathbf{E} \int_{R_{st}} U_{t,uv}^{2n} \, \mathrm{d}\Lambda_{uv} + \mathbf{E} \int_{R_{st}} V_{t,uv}^{2n} \, \mathrm{d}\Lambda_{uv} \right\} < \infty, \tag{4.2}$$

$$\sup_{s,t} \left\{ \mathbf{E} \int_{R_{st}} \tilde{U}_{s,ab}^{2n} \, \mathrm{d}\Lambda_{ab} + \mathbf{E} \int_{R_{st}} \tilde{V}_{s,ab}^{2n} \, \mathrm{d}\Lambda_{ab} \right\} < \infty. \tag{4.3}$$

As shown in the following theorem, a Novikov type condition ensures that L is a martingale.

Theorem 4.2. *Supppose H satisfies the following condition:*

$$\mathbf{E}\left\{ \exp\left(\tfrac{1}{2} \int_{R_{z_0}} H_z^2 \, \mathrm{d}\Lambda_z \right) \right\} < \infty, \tag{4.4}$$

then L is a martingale.

Proof. As can be seen from the given condition, H belongs to $\mathscr{H}_0^2(\mathscr{F})$. Therefore, by the one-parameter Novikov Theorem [12] L is both a 1- and 2-martingale.

Corollary 4.3. *If H satisfies the following condition on R_{z_0}:*

$$\mathbf{E}\left\{ \exp\left(18 \int_{R_{z_0}} H_z^2 \, \mathrm{d}\Lambda_z \right) \right\} < \infty, \tag{4.5}$$

then L is a martingale and $\mathbf{E}(L^{*3}) < \infty$.

Proof. Condition (4.5) implies (4.4), therefore L is a martingale. We have

$$L_z^3 = \exp\{(3H \cdot Y)_z - 9(H^2 \cdot \Lambda)_z\} \exp\{15/2(H^2 \cdot \Lambda)_z\};$$

then

$$\mathbf{E}(L_z^3) \le \{\mathbf{E}(\exp(6H \cdot Y)_z - \tfrac{1}{2}(36H^2 \cdot \Lambda)_z)\}^{1/2} \{\mathbf{E}(\exp 15(H^2 \cdot \Lambda)_{z_0})\}^{1/2}.$$

According to condition (4.4) applied to $6H$, the first factor on the right-side equals 1 and the second is finite according to (4.5). Then $\mathbf{E}(L_z^3)$ is bounded and, by the Cairoli–Doob maximal inequalities, $\mathbf{E}(L^{*3}) < \infty$.

Proposition 4.4. *Let X be a 4-integrable semimartingale and let H satisfy condition (4.5). Then LX is a square-integrable semimartingale of (B, Y).*

Proof. We have, for any z in R_{z_0},

$$\mathbf{E}(L_z^2X_z^2) = \mathbf{E_Q}(L_zX_z^2) \leq [\mathbf{E_Q}(L_z^2)]^{1/2}[\mathbf{E}(X_z^4)]^{1/2}$$

$$\leq [\mathbf{E}(L^{*3})]^{1/2}[\sup_z \mathbf{E}(X_z^4)]^{1/2}.$$

Therefore, $\sup_z \mathbf{E}[(LX)_z^2] < \infty$.

On the other hand, the given conditions on θ, ϕ imply that $L\theta, L\phi$ belong to $\mathscr{H}_0^2(\mathscr{F})$ and processes defined by $L_{y\otimes x}f_{x,y}$, $L_{y\otimes x}g_{x,y}$, with (x, y) in $R_{z_0}^2$, are elements of $\mathscr{H}^2(\mathscr{F})$. From the expression

$$L_{ut}V_{t,uv} = L_{uv}\theta_{uv} + \int_{R_{st}} I_{uv,ab}L_{ub}(g_{uv,ab}\,\mathrm{d}B_{ab} + V_{b,uv}H_{ab}\,\mathrm{d}Y_{ab})$$

and the above conditions, we deduce that the process

$$(I_{uv,ab}L_{ub}V_{b,uv}H_{ab};\,(u, v), (a, b) \in R_{z_0})$$

is necessarily an element of $\mathscr{H}^2(\mathscr{F})$. By a similar argument one can prove that the terms of (2.20) represented by the products $L.U.$, $L.\tilde{V}.$ and $L.\tilde{U}.$ are also elements of $\mathscr{H}^2(\mathscr{F})$. It then follows that in the expressions (2.20) of LX, the local martingale part is a square-integrable martingale. Therefore each term belongs to adequate spaces $\mathscr{H}^2(\mathscr{F})$.

We see that under the conditions of Proposition 4.4, the unnormalized filtering equations hold. But there are cases in which condition (4.5) may not be satisfied and yet the unnormalized filtering equations may be valid. Here we deal with one of these cases.

We first widen the condition under which L is a martingale. The following theorem was already stated without proof in [4]. We give here a detailed proof, since parts of it will be used in the subsequent theorems.

Theorem 4.5. *Suppose H satisfies the following condition:*

$$\exists\, \varepsilon, \delta > 0 \text{ such that } \forall z \in R_{z_0}, \quad \mathbf{E}(\exp \varepsilon H_z^2) < \delta. \tag{4.6}$$

Then L is a martingale.

Proof. Let $0 = s_1 < s_2 < \cdots < s_{n+1} = s_0$ be a partition of $[0, s_0]$ such that $t_0(s_{i+1} - s_i) \leq \frac{1}{18}\varepsilon$ for $i = 1, \ldots, n$. We then put

$$\Delta_i =](s_i, 0), (s_{i+1}, t_0)], \quad I^i = I\{s_i < s \leq s_{i+1}\} \quad \text{and} \quad |\Delta_i| = t_0(s_{i+1} - s_i).$$

We first notice that condition (4.6) implies that $\mathbf{E}(H_z^{2n})$ is bounded on R_{z_0} for any $n \in N^*$ and that condition (4.5) is fulfilled on every Δ_i.

Now, we put

$$\tilde{L}^i_{st}=\exp\left\{\int_{R_{st}} I^i_u H_{uv}\,\mathrm{d}Y_{uv}-\tfrac{1}{2}\int_{R_{st}} I^i_u H^2_{uv}\,\mathrm{d}A_{uv}\right\} \tag{4.7}$$

and

$$L^i_{st}=L^{i-1}_{st}\tilde{L}^i_{st}\quad \text{for } i=2,\ldots,n,\ \text{with } L^1_{st}=\tilde{L}^1_{st}. \tag{4.8}$$

We notice that

$$L^{i-1}_{st}=L_{st}\ \text{for } s\leqslant s_i \quad\text{and}\quad L_{st}=\prod_{i=1}^{n}\tilde{L}^i_{st}.$$

Let us suppose that L^{i-1} is an $(\mathcal{F},\mathbf{P})$-martingale and define a probability $\mathbf{P}^{i-1}$ by $\mathrm{d}\mathbf{P}^{i-1}=L^{i-1}\,\mathrm{d}\mathbf{P}$. Then let W^i be defined by

$$W^i_{st}=Y_{st}-\int_{R_{st}}\left(\sum_{j=1}^{i-1} I^j_u H_{uv}\right)\mathrm{d}A_{uv}. \tag{4.9}$$

According to our Girsanov Theorem 1.2, (B, W^i) is an $(\mathcal{F},\mathbf{P}^{i-1})$-Brownian martingale and, as in Proposition 2.1, its natural filtration, $\mathcal{F}^{B,W^i}$, coincides with $\mathcal{F}$. Therefore, (B, W^i) is an $(\mathcal{F},\mathbf{P}^{i-1})$-Brownian sheet. Using Proposition 1.3, we see that, with respect to W^i, $\mathcal{F}$, $\mathbf{P}^{i-1}$, $\tilde{L}^i$ may be written as follows:

$$\tilde{L}^i=\exp\{I^iH\cdot W^i-\tfrac{1}{2}(I^iH)^2\cdot A\}. \tag{4.10}$$

Corollary 4.3 applies to $\tilde{L}^i$ with $\mathbf{P}$, Y, H, replaced by $\mathbf{P}^{i-1}$, W^i, I^iH, respectively. Since H is $\mathcal{F}^B$-measurable, we have

$$\mathbf{E}_{\mathbf{P}^{i-1}}\left(\exp 18\int_{R_{z_0}} I^i_z H^2_z\,\mathrm{d}A_z\right)=\mathbf{E}_{\mathbf{P}}\left(\exp 18\int_{\Delta_i} H^2_z\,\mathrm{d}A_z\right)<\infty.$$

Therefore, $\tilde{L}^i$ is an $(\mathcal{F},\mathbf{P}^{i-1})$-martingale such that $\mathbf{E}_{\mathbf{P}^{i-1}}(\tilde{L}^{i*})^3$ is finite. Consequently, according to (4.8), L^i is an $(\mathcal{F},\mathbf{P})$-martingale and a recurrence on i leads to the conclusion.

Theorem 4.6. *Suppose that X is a 4-integrable semimartingale and H satisfies the assumptions of Theorem 4.5. Then the unnormalized horizontal and vertical filtering equations are valid.*

Proof. We use the construction and notations of the proof of Theorem 4.5. From (2.21) we can deduce the following equation for $s\in\,]s_i, s_{i+1}]$:

$$\begin{aligned}(LX)_{st}-(LX)_{s_it}=L^{i-1}_{s_it}\bigg\{&\int_{R_{st}} I^i_u\tilde{L}^i_{ut}V_{t,uv}\,\mathrm{d}A_{uv}\\ &+\int_{R_{st}} I^i_u\tilde{L}^i_{ut}H_{uv}\,\mathrm{d}Y_{uv}+\int_{R_{st}} I^i_u\tilde{L}^i_{ut}U_{t,uv}\,\mathrm{d}B_{uv}\bigg\}.\end{aligned} \tag{4.11}$$

For an $\mathscr{F}_z$-measurable and $\mathbf{P}^{i-1}$-integrable r.v. Z, we denote by $\hat{Z}^i$ the version of $\mathbf{E}_{\mathbf{P}^{i-1}}(Z/\mathscr{G}_z)$ defined by

$$\hat{Z}^i = k(L_z^{i-1}Z)/k(L_z^{i-1}). \tag{4.12}$$

Formula (4.11) gives, for $s \in]s_i, s_{i+1}]$,

$$k(LX)_{st} - k(LX)_{s_it} = k(L_{s_it}^{i-1})\left\{ \widehat{\int_{R_{st}} I_u^i \tilde{L}_{ut}^i V_{t,uv}\, \mathrm{d}A_{uv}}^{\,i} + \widehat{\int_{R_{st}} I_u^i \tilde{L}_{ut}^i H_{uv}\, \mathrm{d}Y_{uv}}^{\,i} + \widehat{\int_{R_{st}} I_u^i \tilde{L}_{ut}^i U_{t,uv}\, \mathrm{d}B_{uv}}^{\,i} \right\} \text{ a.s.} \tag{4.13}$$

As we shall indicate below, this equality becomes

$$k(LX)_{st} - k(LX)_{s_it} = = k(L_{s_it}^{i-1})\left\{ \int_{R_{st}} I_u^i \widehat{\tilde{L}_{ut}^i V_{t,uv}}^{\,i}\, \mathrm{d}A_{uv} + \int_{R_{st}} I_u^i \widehat{\tilde{L}_{ut}^i X_{ut} H_{uv}}^{\,i}\, \mathrm{d}Y_{uv} \right\} \text{ a.s.} \tag{4.14}$$

Then the right-hand side of (4.14) can be written as

$$\int_{R_{st}} I_u^i k(L_{s_it}^{i-1}) \widehat{\tilde{L}_{ut}^i V_{t,uv}}^{\,i}\, \mathrm{d}A_{uv} + \int_{R_{st}} I_u^i k(L_{s_it}^{i-1}) \widehat{\tilde{L}_{ut}^i X_{ut} H_{uv}}^{\,i}\, \mathrm{d}Y_{uv}.$$

But, since $L_{s_it}^{i-1} = L_{ut}^{i-1}$ for $u \in]s_i, s_{i+1}]$, this is equal to

$$\int_{R_{st}} I_u^i k(L_{ut}^{i-1}) \widehat{\tilde{L}_{ut}^i V_{t,uv}}^{\,i}\, \mathrm{d}A_{uv} + \int_{R_{st}} I_u^i k(L_{ut}^{i-1}) \widehat{\tilde{L}_{ut}^i X_{ut} H_{uv}}^{\,i}\, \mathrm{d}Y_{uv},$$

and, by (4.12), to

$$\int_{R_{st}} I_u^i k(L_{ut}^{i-1} \tilde{L}_{ut}^i V_{t,uv})\, \mathrm{d}A_{uv} + \int_{R_{st}} I_u^i k(L_{ut}^{i-1} \tilde{L}_{ut}^i X_{ut} H_{uv})\, \mathrm{d}Y_{uv} =$$

$$= \int_{R_{st}} I_u^i (KL)_{ut}(V_{t,uv}/ut)\, \mathrm{d}A_{uv} + \int_{R_{st}} I_u^i (KL)_{ut}(X_{ut}H_{uv}/u,t)\, \mathrm{d}Y_{uv}.$$

By summing with respect to i we obtain the unnormalized horizontal filtering equation (2.21). The unnormalized vertical filtering equation (2.22) can be established in the same way.

Now, it remains to justify the passage from (4.13) to (4.14). First, we notice that $\tilde{L}^i X$ is a square-integrable semimartingale of (B, W^i) under $\mathbf{P}^{i-1}$. Therefore, the integrands of the right-hand side of (4.13) are in $\mathscr{H}_1^2(\mathscr{F}, \mathbf{P}^{i-1})$. On the other hand, for any fixed t and for all s, s' such that $s_i < s \leq s' \leq s_{i+1}$ we have

$$\forall A \in \mathscr{F}_{st}: \quad \mathbf{P}^{i-1}(A/\mathscr{G}_{st}) = k(L_{st}^{i-1} I(A))/k(L_{st}^{i-1})$$
$$= k(L_{s't}^{i-1} I(A))/k(L_{s't}^{i-1}) = \mathbf{P}^{i-1}(A/\mathscr{G}_{s't}) \tag{4.15}$$

because $L_{s't}^{i-1} = L_{st}^{i-1} = L_{s_i t}$.

This property is the same as the one used in [1] and [15] for the construction of a projection theorem similar to Proposition 2.3. The passage from (4.13) to (4.14) can be proved by adapting the proof of [13] or [15] to the situation considered here.

Theorem 4.7. *Let X be an $\mathscr{E}$-integrable semimartingale and H satisfy the assumptions of Theorem 4.5. Then, the unnormalized diagonal filtering equation holds.*

Proof. We start from the unnormalized horizontal filtering equation

$$k(LX)_{st} = \int_{R_{st}} k(L_{ut} V_{t,uv})\, \mathrm{d}A_{uv} + \int_{R_{st}} k(L_{ut} X_{ut} H_{uv})\, \mathrm{d}Y_{uv} \tag{4.16}$$

and we replace in this equation the processes

$$(L_{ut} V_{t,uv}, (u, v) \in [0, s_0] \times [0, t]) \quad \text{and} \quad (L_{ut} X_{ut} H_{uv}, (u, v) \in [0, s_0] \times [0, t])$$

by their expressions deduced from their unnormalized vertical filtering equations.

By the Ito formula [23] we have

$$L_{ut} V_{t,uv} = L_{uv}\theta_{uv} + \int_{R_{st}} I_{uv,ab} L_{ub}(g_{uv,ab}\, \mathrm{d}B_{ab} + V_{b,uv} H_{ab}\, \mathrm{d}Y_{ab}).$$

By Theorem 4.6 this gives

$$k(L_{ut} V_{t,uv}) = k(L_{uv}\theta_{uv}) + \int_{]0,u]\times]v,t]} k(L_{ub} V_{b,uv} H_{ab})\, \mathrm{d}Y_{ab}. \tag{4.17}$$

We obtain in the same way

$$k(L_{ut} X_{ut} H_{uv}) = k(L_{uv} X_{uv} H_{uv})$$
$$+ \int_{]0,u]\times]v,t]} \{k(L_{ub} \tilde{V}_{u,ab} H_{uv})\, \mathrm{d}A_{ab} + k(L_{ub} X_{ub} H_{ab} H_{uv})\, \mathrm{d}Y_{ab}\}. \tag{4.18}$$

Finally, by substituting (4.17) and (4.18) into (4.16) we obtain the unnormalized filtering equation (2.23).

To conclude this section we summarize the main results in the following theorem.

Theorem 4.8. *Let X be a $2n$-integrable semimartingale of the Brownian sheet B (with*

$n \in N^*$) and let H be a $\mathcal{B}$-adapted process such that

$$\int_{R_{z0}} \mathbf{E}(H_z^{2m})\,\mathrm{d}A_z < \infty \quad (\textit{for some } m \geq 1).$$

Consider the following situations S_i for $i = 1, 2, 3$:

- *S_1: H is bounded,*
- *S_2: $\mathbf{E}\left(\exp\left\{18 \int_{R_{z0}} H_z^2\,\mathrm{d}A_z\right\}\right) < \infty$,*
- *S_3: $\exists \varepsilon, \delta > 0$ such that $\mathbf{E}(\exp \varepsilon H_z^2) < \delta$.*

Then the horizontal and vertical filtering equations (3.5) *and* (3.6) *hold if* ($i = 1, n \geq 2$) *or* ($i = 2, n \geq 2, m \geq 2$) *or* ($i = 3, n \geq 2$).

In addition, the diagonal filtering equation (3.12) *holds if* ($i = 1, n \geq 2$) *or* ($i = 2, n \geq 4, m \geq 4$) *or* ($i = 3, n \geq 4$).

Proof. The proof is a simple matter of verification.

In [11] the filtering equations we obtained are related to the case ($i = 1, n \geq 2$). The Gaussian model considered in the next section is covered by the case ($i = 3, n \geq 4$).

5. Case of diffusion processes

The Gaussian linear model mentioned above is a particular case of a more general model in which the signal X is the solution of a set of stochastic differential equations.

Let M be the process defined by

$$M = G \cdot B \tag{5.1}$$

where B is a Brownian sheet and G a nonnegative nonrandom square-integrable function on R_{z0}, and let X be a continuous process satisfying the following equations:

$$X_{st} = \int_0^s A(u, t, X_{ut})\,\mathrm{d}u + \int_0^s D(u, t, X_{ut}) M(\mathrm{d}u, t), \tag{5.2}$$

$$X_{st} = \int_0^t \tilde{A}(s, v, X_{sv})\,\mathrm{d}v + \int_0^t D(s, v, X_{sv}) M(s, \mathrm{d}v) \tag{5.3}$$

where A, $\tilde{A}$ and D are continuous nonrandom functions on $R_{z0} \times \mathbb{R}$.

It is shown in [10] that under certain regularity conditions on A, $\tilde{A}$, D, such a process X satisfies an equation of the following type:

$$X_{st} = \int_{R_{st}} \theta_{uv}\, \mathrm{d}\Lambda_{uv} + \int_{R_{st}} g(u, v, X_{uv}) M(\mathrm{d}u, \mathrm{d}v)$$
$$+ \int_{R_{st}} \phi(u, v, X_{uv}) M(u, \mathrm{d}v)\, \mathrm{d}u + \int_{R_{st}} \tilde{\phi}(u, v, X_{uv}) M(\mathrm{d}u, v)\, \mathrm{d}v$$
$$+ \int_{R_{st}} \Psi(u, v, X_{uv}) M(\mathrm{d}u, v) M(u, \mathrm{d}v) \tag{5.4}$$

a [illegible] sses a Markov property as defined in [9]. In this case X is called a *diffusion p[illegible]*

[illegible] the same method as for a semimartingale, one can obtain the filtering equations for such a process. These equations and the corresponding Riccati-type equations have expressions deduced from the general case by the following formal substitutions:

$$U_{t,uv} = D(u, t, X_{ut}) G_{uv}, \qquad \tilde{U}_{s,ab} = D(s, b, X_{sb}) G_{ab},$$
$$\int_0^t V_{t,uv}\, \mathrm{d}v = A(u, t, X_{ut}), \qquad \int_0^s \tilde{V}_{s,ab}\, \mathrm{d}a = \tilde{A}(s, b, X_{sb}).$$

Now, let X be a Gaussian Markov process [8] defined by

$$X_z = D_z M_z \tag{5.5}$$

where M is the Gaussian strong $\mathscr{B}$-martingale (5.1) and D a strictly positive nonrandom real function on R_{z_0} with continuous partial derivatives $\partial D_{st}/\partial s$, $\partial D_{st}/\partial t$ and $\partial^2 D_{st}/\partial s\, \partial t$. Then X is a diffusion process satisfying the following equations:

$$X_{st} = \int_0^s \frac{\partial D_{ut}}{\partial u} D_{ut}^{-1} X_{ut}\, \mathrm{d}u + \int_0^s D_{ut} M(\mathrm{d}u, t), \tag{5.6}$$

$$X_{st} = \int_0^t \frac{\partial D_{sv}}{\partial v} D_{sv}^{-1} X_{sv}\, \mathrm{d}v + \int_0^t D_{sv} M(s, \mathrm{d}v), \tag{5.7}$$

$$X_{st} = \int_{R_{st}} \frac{\partial^2 D_{uv}}{\partial u\, \partial v} D_{uv}^{-1} X_{uv}\, \mathrm{d}\Lambda_{uv} + \int_{R_{st}} D_{uv} M(\mathrm{d}u, \mathrm{d}v)$$
$$+ \int_{R_{st}} \frac{\partial D_{uv}}{\partial u} M(u, \mathrm{d}v)\, \mathrm{d}u + \int_{R_{st}} \frac{\partial D_{uv}}{\partial v} M(\mathrm{d}u, v)\, \mathrm{d}v. \tag{5.8}$$

Suppose that process H is defined by

$$H_{st} = h_{st} X_{st} \tag{5.9}$$

where h is a nonrandom continuous function on R_{z_0}.

Since D is a nonrandom function, one can deduce the filtering equations for X from those of $M=D^{-1}X$. Thus (3.5) and (3.6) provide the following expressions:

$$(X_{s\tau}/st)=D_{s\tau}\int_{R_{st}} D_{u\tau}^{-1}\rho_{ut}(X_{u\tau},X_{uv})h_{uv}\,\mathrm{d}\nu_{uv}^{1,t} \quad \text{for } \tau\leq t, \tag{5.10}$$

$$(X_{\sigma t}/st)=D_{\sigma t}\int_{R_{st}} D_{\sigma b}^{-1}\rho_{sb}(X_{\sigma b},X_{ab})h_{ab}\,\mathrm{d}\nu_{ab}^{2,s} \quad \text{for } \sigma\leq s. \tag{5.11}$$

The last term of diagonal filtering equation (3.9) vanishes in this case. In order to simplify the notation we write this in the following formal differential form for M_ζ with $\zeta\in\partial R_{st}$ (cf. (3.7)):

$$\begin{aligned}\mathrm{d}[D_\zeta^{-1}(X_\zeta/st)]&=D_\zeta^{-1}\rho_{st}(X_\zeta,X_{st})h_{st}\,\mathrm{d}\nu_{st}^0\\&\quad+\int_{v=0}^{t} d_t[D_\zeta^{-1}\rho_{st}(X_\zeta,X_{sv})h_{sv}]\,\mathrm{d}\nu_{sv}^{1,t}\\&\quad+\int_{a=0}^{s} d_s[D_\zeta^{-1}\rho_{st}(X_\zeta,X_{at})h_{at}]\,\mathrm{d}\nu_{at}^{2,s}.\end{aligned}\tag{5.12}$$

Conditional covariance functions ρ in (5.10)–(5.12) are entirely determined by the following generalized Riccati equations:

$$\begin{aligned}\frac{\partial}{\partial s}\rho_{st}(X_{st_1},X_{st_2})&=\rho_{st}(X_{st_1},X_{st_2})\left[\frac{\partial D_{st_1}}{\partial s}D_{st_1}^{-1}+\frac{\partial D_{st_2}}{\partial s}D_{st_2}^{-1}\right]\\&\quad+D_{st_1}D_{st_2}\int_0^{t_1\wedge t_2}G_{sv}^2\,\mathrm{d}v\\&\quad-\int_0^t\rho_{st}(X_{st_1},X_{sv})h_{sv}^2\rho_{st}(X_{st_2},X_{sv})\,\mathrm{d}v \quad \text{for } t_1\leq t,\ t_2\leq t,\end{aligned}\tag{5.13}$$

$$\begin{aligned}\frac{\partial}{\partial s}\rho_{st}(X_{\sigma t},X_{s\tau})&=\rho_{st}(X_{\sigma t},X_{s\tau})\frac{\partial D_{s\tau}}{\partial s}D_{s\tau}^{-1}\\&\quad-\int_0^t\rho_{st}(X_{s\tau},X_{sv})h_{sv}^2\rho_{st}(X_{\sigma t},X_{sv})\,\mathrm{d}v \quad \text{for } \sigma<s,\ \tau\leq t\end{aligned}\tag{5.14}$$

with two other equations written by permuting the roles of s and t in (5.13) and (5.14).

Linear filtering equations and corresponding Riccati equations for Gaussian fields were obtained in [16] (and lateral filtering equations in [17]) from the representation of $\mathcal{G}$-weak martingales and $\mathcal{G}$-1 or 2-martingales under probability $\mathbf{Q}$. The lateral filtering equations (5.12) and (5.13) and corresponding Riccati equations for the Gaussian Markov process (5.5) were obtained in [8] by the linear filtering method of Hilbert space-valued processes and diagonal filtering equation by a geometric approach. The results presented here in the nonlinear case thus generalize those of the already known linear filtering problem.

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